Exactly solvable model with an absorbing state and multiplicative colored Gaussian noise

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We study the temporal evolution of a system that has an absorbing state and that is driven by colored Gaussian noise, whose amplitude depends on the system state x as $|x|^{\alpha}$. Exact, analytical expressions for the probability density functions of the system and of the absorption time are derived. We also calculate numerical characteristics of the probability density functions, namely, the fractional moments of the system and the mean absorption time, and analyze the role of the functional form of the noise correlation function.

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I. INTRODUCTION

Many systems and models in physics, chemistry, biology, and other fields possess absorbing states. Such states can be reached by the dynamics of the system, but once the system attains an absorbing state, the dynamics does not allow it to escape and the system is trapped in that state. Systems with absorbing states have attracted particular attention over the last decade, since they can display nonequilibrium phase transitions between active and inactive phases [1–3]. Well-studied examples include the spreading of epidemics [4,5], reaction-diffusion models of surface catalysis [6,7], transport in random media [8,9], forest fires [10], surface growth [11], and contact processes with aging [12].

The simplest nonequilibrium situations are models or systems without spatial structure. Examples are well-stirred autocatalytic chemical reactions, such as the Schlögl model [13], lumped models in population biology, such as the logistic or Verhulst model [14], or an overdamped particle in a potential subjected to a random driving force [3]. Such systems can display absorbing states, and their dynamics can be described by a Langevin equation,

$$\dot{x}(t)=F(x(t))+g(x(t))f(t), \tag{1.1}$$

where $F(x)$ and $g(x)$ are deterministic functions, and $f(t)$ is a stationary Gaussian noise with zero mean and arbitrary correlation function $R(u)$ [$R(0)>0$, $R(\infty)=0$]. Equation (1.1) possesses an absorbing state $\tilde{x}$, if that state can be reached dynamically and if both the systematic force and the random driving force vanish in that state, i.e., if $F(\tilde{x})=0$ and $g(\tilde{x})=0$. Clearly, only systems with multiplicative noise, i.e., $g(x)\not\equiv\text{const}$, can display absorbing states that are an intrinsic property of the dynamics.

We study the statistical properties of a nonlinear system that belongs to the class defined by Eq. (1.1). It is described by the Langevin equation

$$\dot{x}(t)=-\kappa x(t)+|x(t)|^{\alpha}f(t), \tag{1.2}$$

and has the advantage of being exactly solvable. The state $x=0$ can be an absorbing state for this system only if $\alpha>0$. For $\alpha\geqslant 1$, this state cannot be reached in a finite amount of time, and we restrict our consideration therefore to the interval $0<\alpha<1$. Most commonly encountered in applications is the value $\alpha=1/2$. Multiplicative noise of this type occurs in models of lasers [15], and in models of chemical reactions and epidemics [16–19]. The latter belong to the universality class that can be represented by the Langevin equation of Reggeon field theory. The spatially homogeneous version of that equation coincides with Eq. (1.2) for small x. In addition to considering the Langevin equation (1.2) as a model in its own right, it can also be thought of as an approximation that retains only the lowest order term of the systematic force $F(x)$ and the multiplicative noise term $g(x)$ in an expansion around the absorbing state. For this reason, we consider not only the case $\kappa>0$, where the systematic force in the absence of noise drives the system toward the absorbing state, i.e., $x=0$ is deterministically linearly stable, but also the case $\kappa<0$, which corresponds to a deterministically linearly unstable steady state at $x=0$.

As discussed in a previous paper [20], Eq. (1.2) has the interesting feature that for $0<\alpha<1$ the solution is not unique at $x=0$. There are two solutions that pass through zero, one for which $x=0$ is a regular point and one for which $x=0$ is an absorbing state. In Ref. [20], we studied the former type of solution, which coincides with that of the Langevin equation

$$[\dot{x}(t)+\kappa x(t)]|x(t)|^{-\alpha}=f(t) \tag{1.3}$$

for all times $t\geqslant 0$. Specifically, we found the univariate and bivariate probability density functions (PDFs), the fractional moments, the correlation function, and the fractal dimension of the solution. As mentioned above, in this paper we study the statistical properties of the system (1.2) with an absorbing state at $x=0$. In that case, for the initial condition $x(0)=x_0>0$, the solution of Eq. (1.2) coincides with the solution of Eq. (1.3) only for $0\leqslant t\leqslant t_0$. For $t>t_0$, the system (1.2) remains trapped in the state $x=0$, i.e., $x(t)\equiv 0$, whereas sys-

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tem (1.3) has a nonzero solution [20]. Here $t_0 \in (0,\infty]$ is the absorption time. It is the instant when $x(t)=0$ for the first time, or, in other words, t_0 is the so-called first-passage time. At the random instant of time $t=t_0$, the system (1.2) becomes trapped in the dynamically completely inactive state $x=0$.

We use the results of Ref. [20] to derive exact, analytical expressions for the PDF of the system (1.2) with $x=0$ an absorbing state, as well as the absorption time probability density. We find that $\kappa=0$ is the critical value. For κ positive, absorption is certain, while for κ negative, the system remains in the active state with nonzero probability even as time goes to infinity. At the critical value of κ, either various types of diffusive behavior or stochastic localization occur, depending on the correlation function of the noise. Only in the former case is absorption the ultimate fate of the system. Survival is possible in the latter case.

The paper is organized as follows. In Sec. II we find the PDF for the solution of Eq. (1.2). We derive the exact expressions for the fractional moments of $x(t)$ and calculate their asymptotic behavior in Sec. III. In Sec. IV we find the PDF for the absorption time. We calculate the mean absorption time in Sec. V. We also determine its behavior as the initial state of the system approaches either the absorbing state or the noise intensity goes to infinity as well as the limit of the initial state going to infinity or the intensity of the noise going to zero. We discuss extensions of our results in Sec. VI.

II. PROBABILITY DENSITY FUNCTION

A. PDF for Eq. (1.3)

For the convenience of the reader, we briefly summarize the main results of Ref. [20]. To find the PDF $P(x,t)$ for the solution of Eq. (1.2) with $x=0$ an absorbing state, we obtain first the equation for the PDF $P_x(x,t)$ of the solution of Eq. (1.3). According to Ref. [20], the solution of the latter equation has the form

$$x(t)=[x_0^{1-\alpha}e^{-\omega t}+q(t)]|x_0^{1-\alpha}e^{-\omega t}+q(t)|^{\alpha/(1-\alpha)}, \tag{2.1}$$

where $\omega=(1-\alpha)\kappa$, and

$$q(t)=(1-\alpha)\int_0^t dt' e^{-\omega(t-t')}f(t'). \tag{2.2}$$

Since

$$q(t)=x(t)|x(t)|^{-\alpha}-x_0^{1-\alpha}e^{-\omega t}, \tag{2.3}$$

a one-to-one correspondence exists between $x(t)$ and $q(t)$. If $P_q(q,t)$ is the probability density that $q(t)=q$, then $P_x(x,t)dx=P_q(q,t)dq$ and we obtain

$$P_x(x,t)=\frac{1-\alpha}{|x|^\alpha}P_q\left(\frac{x}{|x|^\alpha}-x_0^{1-\alpha}e^{-\omega t},t\right). \tag{2.4}$$

If $f(t)$ is a Gaussian noise, then $q(t)$ is also a Gaussian process, and Eq. (2.4) is reduced to the power-normal distribution [20]

$$P_x(x,t)=\frac{1-\alpha}{\sqrt{2\pi}\sigma_q(t)|x|^\alpha}\exp\left\{-\frac{1}{2\sigma_q^2(t)}\left(\frac{x}{|x|^\alpha}-x_0^{1-\alpha}e^{-\omega t}\right)^2\right\} \tag{2.5}$$

$[x\in(-\infty,\infty)]$. Here $\sigma_q^2(t)=\langle q^2(t)\rangle$ is the dispersion of $q(t)[\langle\rangle$ denotes averaging with respect to the noise $f(t)]$, which is given by

$$\sigma_q^2(t)=2(1-\alpha)^2\frac{e^{-\omega t}}{\omega}\int_0^t du\; R(u)\sinh[\omega(t-u)]. \tag{2.6}$$

One can verify that $P_x(x,t)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}P_x(x,t)=\frac{\partial}{\partial x}[\kappa x-\Delta_\omega(t)\alpha x|x|^{2(\alpha-1)}]P_x(x,t)+\Delta_\omega(t)\frac{\partial^2}{\partial x^2}|x|^{2\alpha}P_x(x,t), \tag{2.7}$$

where

$$\Delta_\omega(t)=\int_0^t du\; R(u)e^{-\omega u}=\frac{\sigma_q(t)\dot{\sigma}_q(t)+\omega\sigma_q^2(t)}{(1-\alpha)^2}. \tag{2.8}$$

Specifically, if $f(t)$ is Gaussian white noise, then $R(u)=2\Delta\delta(u)$ (Δ is the white noise intensity), and Eq. (2.8) yields $\Delta_\omega(t)=\Delta$. The solution $x(t)$ is a Markovian diffusion process, and Eq. (2.7) corresponds to the Stratonovich interpretation [21] of Eq. (1.2). To avoid any misunderstandings, we emphasize that for colored Gaussian noise $f(t)$ the random process $x(t)$ is not Markovian, in spite of the fact that $P_x(x,t)$ obeys the Fokker-Planck equation (2.7).

B. PDF for Eq. (1.2)

Recall that the solution of Eq. (1.2) with an absorbing state at $x=0$ coincides with the solution of Eq. (1.3) up to the random absorption time t_0. Therefore, the PDF $P(x,t)$ of Eq. (1.2) can be obtained in the following way. Let $W(x,t)(0\leqslant x\leqslant\infty)$ be the solution of Eq. (2.7) that satisfies the absorbing boundary condition $W(0,t)=0$. As for Markovian diffusion processes [22–24], the PDF $P(x,t)$ is then given by

$$P(x,t)=W(x,t)+A(t)\delta(x), \tag{2.9}$$

where

$$A(t)=1-\int_0^\infty dx W(x,t) \tag{2.10}$$

is the probability that at time t the system described by Eq. (1.2) is in the absorbing state $x=0$. Using the method of images (see, for example, Ref. [25]) we can express $W(x,t)$ in terms of $P_x(x,t)$,

$$W(x,t)=P_x(x,t)-P_x(-x,t), \tag{2.11}$$

and from Eqs. (2.5) and (2.10) we find

$$A(t)=\text{erfc}[a(t)/\sqrt{2}]. \tag{2.12}$$

Here $\text{erfc}(z)=(2/\sqrt{\pi})\int_z^\infty dt\,\exp(-t^2)$ is the complementary error function, and $a(t)=x_0^{1-\alpha}e^{-\omega t}/\sigma_q(t)$.

It is easily verified from Eq. (2.11) that $W(x,t)$ indeed fulfills the absorbing boundary condition $W(0,t)=0$, but the asymptotic behavior of $W(x,t)$ as $x\to 0$ can be singular. For $t>0$, Eqs. (2.5) and (2.11) lead to the asymptotic formula

$$W(x,t)\sim\sqrt{\frac{2}{\pi}}\frac{(1-\alpha)a(t)}{\sigma_q^2(t)}e^{-a^2(t)/2}x^{1-2\alpha} \tag{2.13}$$

$(x\to 0)$. This implies that $W(x,t)\to 0$ for $0<\alpha<1/2$, and $W(x,t)\to\infty$ for $1/2<\alpha<1$. Note that the singularity of $W(x,t)$ for $1/2<\alpha<1$ is an integrable one.

C. Short- and long-time behavior of the absorption probability

The absorption probability $A(t)$, the probability that at time t the system is in the dynamically inactive state $x=0$, is an important quantity for characterizing the temporal behavior of the system (1.2). We now analyze the short- and long-time behavior of this quantity. Since $x(0)=x_0>0$, we expect the absorption probability to approach zero as t goes to zero. The ultimate fate of the system is described by the long-time behavior of $A(t)$. If $\kappa>0$ $(\omega>0)$, i.e., the systematic force drives the system toward the absorbing state and $x=0$ is a deterministically stable state, then we expect the system to become trapped eventually in the absorbing state as time goes to infinity. On the other hand, if $\kappa<0$ $(\omega<0)$, i.e., the systematic force drives the system away from zero and the absorbing state is deterministically unstable, then we expect the opposing effects of the systematic force and the random driving force to render the eventual trapping of the system less certain.

It follows from Eq. (2.12) that the asymptotic behavior of $A(t)$ as $t\to 0$ and $t\to\infty$ is determined by the asymptotic behavior of $a(t)$. We consider the case, frequently encountered in applications, that the leading asymptotic term of the correlation function of the colored Gaussian noise $R(u)$ obeys a power law, i.e., $R(u)\sim c_\alpha u^{-\beta}$ as $u\to 0$ ($c_\alpha>0$, $0\leqslant\beta<1$). Then Eq. (2.6) yields

$$\sigma_q^2(t)\sim\frac{2c_\alpha(1-\alpha)^2}{(1-\beta)(2-\beta)}t^{2-\beta}\quad(t\to 0), \tag{2.14}$$

and

$$a(t)\sim\frac{x_0^{1-\alpha}}{1-\alpha}\sqrt{\frac{(1-\beta)(2-\beta)}{2c_\alpha}}\frac{1}{t^{1-\beta/2}}\quad(t\to 0), \tag{2.15}$$

i.e., the parameter $a(t)$ diverges as $t\to 0$. Taking into account that $\text{erfc}(x)\sim\exp(-x^2)/(\sqrt{\pi}x)$ as $x\to\infty$, we obtain from Eq. (2.12) for $t\to 0$ the asymptotic formula

$$A(t)\sim\sqrt{\frac{2}{\pi}}\frac{1}{a(t)}\exp\left(-\frac{a^2(t)}{2}\right), \tag{2.16}$$

where $a(t)$ is given by Eq. (2.15). As expected, $A(t)\to 0$ as $t\to 0$ in agreement with the initial condition $x(0)=x_0\neq 0$.

If $\kappa>0$ $(\omega>0)$, then

$$\sigma_q^2(\infty)=(1-\alpha)^2\frac{1}{\omega}\int_0^\infty du\,R(u)e^{-\omega u} \tag{2.17}$$

[since $R(u)\to 0$ as $u\to\infty$, the condition $\sigma_q(\infty)<\infty$ holds], and $a(t)\sim x_0^{1-\alpha}e^{-\omega t}/\sigma_q(\infty)$ and

$$1-A(t)\sim\sqrt{\frac{2}{\pi}}\frac{x_0^{1-\alpha}}{\sigma_q(\infty)}e^{-\omega t}, \tag{2.18}$$

as $t\to\infty$. So the probability of finding the system in the absorbing state $x=0$ tends to 1 if $t\to\infty$, as expected.

For $\kappa<0$ $(\omega<0)$ and $t\to\infty$, Eq. (2.6) yields

$$\sigma_q^2(t)\sim(1-\alpha)^2\frac{e^{2|\omega|t}}{|\omega|}\int_0^\infty du\,R(u)e^{-|\omega|u}, \tag{2.19}$$

which implies

$$a(\infty)=\frac{x_0^{1-\alpha}}{1-\alpha}\left(\frac{1}{|\omega|}\int_0^\infty du\,R(u)e^{-|\omega|u}\right)^{-1/2}, \tag{2.20}$$

and $A(\infty)=\text{erfc}[a(\infty)/\sqrt{2}]$. As expected, $A(\infty)<1$. The random driving force is not strong enough to overcome the systematic force with probability 1; the system has a nonzero probability of surviving indefinitely in the active state.

Finally, for $\kappa=0$ $(\omega=0)$, Eq. (2.6) is reduced to

$$\sigma_q^2(t)=2(1-\alpha)^2\int_0^t du\,R(u)(t-u). \tag{2.21}$$

If

$$\Delta_0(t)\equiv\int_0^t du\,R(u)=o(1/t)\quad(t\to\infty), \tag{2.22}$$

then $\sigma_q(\infty)<\infty$, i.e., the phenomenon of stochastic localization of $x(t)$ occurs [26] (see also Sec. III), and $A(\infty)<1$. This remarkable result shows that there is a nonzero probability that the system is in the active state as $t\to\infty$. From a physical point of view, the possibility of the system surviving indefinitely is due to the fact that $x(t)$ and $x(t_1)$ are correlated in the case of stochastic localization even for $|t_1-t|\to\infty$ [20]. Stochastic localization occurs if the noise intensity $R\equiv\Delta_0(\infty)$ vanishes, i.e., if contributions from regions of positive and negative correlations in the noise $f(t)$ cancel each other out. It is this balance in the random driving force itself that allows the system to survive in the active state with a nonzero probability. Otherwise, i.e., if either $0<R\leqslant\infty$ or $R=0$ and condition (2.22) does not hold, we have $\sigma_q(\infty)=\infty$ and $A(\infty)=1$.

Our results show that sublinear multiplicative colored Gaussian noise does not change the critical value κ_c for a linear restoring force. We find that for $\kappa>0$, $A(\infty)=1$, and for $\kappa<0$, $A(\infty)<1$, i.e., $\kappa_c=0$. The critical situation itself, however, splits into two cases. If the noise intensity R vanishes and condition (2.22) is fulfilled, the system has a chance of ultimate survival, whereas otherwise ultimate trapping occurs with probability 1.

III. FRACTIONAL MOMENTS

In the previous section, we achieved our main goal, namely, to obtain the PDF of the solution $x(t)$ of Eq. (1.2) with an absorbing state at $x=0$, and the absorption probability. In this section, we will consider a more concise description of the system and calculate numerical characteristics of the random process $x(t)$. Moments are of particular interest in applications, and here we consider the fractional moments $m_r(t)(r>0)$ of $x(t)$,

$$m_r(t)=\int_0^\infty dx\, x^r P(x,t). \tag{3.1}$$

Using the integral representation of the Weber parabolic cylinder functions [27]

$$D_{-\mu}(x)=\frac{e^{-x^2/4}}{\Gamma(\mu)}\int_0^\infty dy\, y^{\mu-1}e^{-y^2/2-xy} \quad (\mu>0) \tag{3.2}$$

[$\Gamma(\mu)=\int_0^\infty dy\, y^{\mu-1}e^{-y}$ is the gamma function], we can reduce Eq. (3.1) to the form

$$m_r(t)=\frac{\Gamma(\xi)}{\sqrt{2\pi}}e^{-a^2(t)/4}\sigma_q^{\xi-1}(t)\{D_{-\xi}[-a(t)]-D_{-\xi}[a(t)]\}, \tag{3.3}$$

where $\xi=1+r/(1-\alpha)$.

We now study the long-time behavior of $m_r(t)$. If $\kappa>0$, then using the asymptotic formula

$$D_{-\xi}(-x)-D_{-\xi}(x)\sim\frac{2^{\xi/2+1/2}}{\Gamma(\xi)}\Gamma\left(\frac{\xi+1}{2}\right)x \quad (x\to 0), \tag{3.4}$$

which follows from Eq. (3.2), and taking into account that $a(t)\to 0$ as $t\to\infty$, we obtain from Eq. (3.3)

$$m_r(t)\sim\frac{x_0^{1-\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{\xi+1}{2}\right)2^{\xi/2}\sigma_q^{\xi-2}(\infty)e^{-\omega t} \quad (t\to\infty). \tag{3.5}$$

The fact that $m_r(\infty)=0$ together with $A(\infty)=1$ implies that $P(x,\infty)=\delta(x)$. The stationary probability of the system is entirely concentrated on the absorbing state $x=0$.

If $\kappa<0$, then Eqs. (2.19) and (2.20) yield $\sigma_q(t)\sim x_0^{1-\alpha}e^{|\omega|t}/a(\infty)$, and Eq. (3.3) leads to the asymptotic formula

$$m_r(t)\sim x_0^r\frac{\Gamma(\xi)e^{-a^2(\infty)/4}}{\sqrt{2\pi}a^{\xi-1}(\infty)}\{D_{-\xi}[-a(\infty)]-D_{-\xi}[a(\infty)]\}e^{r|\kappa|t} \tag{3.6}$$

$(t\to\infty)$, where $a(\infty)$ is given by Eq. (2.20). Thus, for $\kappa<0$ all fractional moments diverge as $t\to\infty$; no stationary PDF exists. Recall that $A(\infty)<1$. The fact that all moments go to infinity as $t\to\infty$ has the following implication. If the system avoids being trapped in the state $x=0$, then there is a nonzero probability that $x(t)$ undergoes arbitrarily large excursions as $t\to\infty$.

If $\kappa=0$ and Eq. (2.22) holds, then $\sigma_q(\infty)<\infty$ and all fractional moments $m_r(\infty)$ are finite, i.e., stochastic localization of $x(t)$ occurs. When Eq. (2.22) does not hold, i.e., $\sigma_q(\infty)=\infty$, Eqs. (3.3) and (3.4) yield

$$m_r(t)\sim\frac{x_0^{1-\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{\xi+1}{2}\right)2^{\xi/2}\sigma_q^{\xi-2}(t) \tag{3.7}$$

$(t\to\infty)$. According to this formula, $m_r(\infty)=0$ for $0<r<1-\alpha$, $0<m_r(\infty)<\infty$ for $r=1-\alpha$, and $m_r(\infty)=\infty$ for $r>1-\alpha$. So, although $A(\infty)=1$, all fractional moments $m_r(\infty)$ with $r>1-\alpha$ are infinite. The system will almost surely be trapped in the absorbing state as $t\to\infty$; however, no stationary PDF exists. Even as time goes to infinity, the diffusive motion of the system results in excursions arbitrarily far away from the absorbing state. Specifically, the dispersion of the system state, $\sigma_x^2(t)=\langle x^2(t)\rangle-\langle x(t)\rangle^2$, is given by $\sigma_x^2(t)=m_2(t)-m_1^2(t)$ and it follows from Eq. (3.7) that $\sigma_x^2(t)\sim m_2(t)\to\infty$ as $t\to\infty$. If $\Delta_0(t)\propto t^\eta$ $(-1<\eta<1)$ as $t\to\infty$, then $\sigma_q^2(t)\propto t^{1+\eta}$, and therefore $\sigma_x^2(t)\propto t^{(1+\eta)(1+\alpha)/(1-\alpha)}$. The last relation shows that normal diffusion [diffusion with $\sigma_x^2(t)\propto t$ as $t\to\infty$], subdiffusion (diffusion slower then the normal), and superdiffusion (diffusion faster then the normal) of $x(t)$ take place, if $\eta=-2\alpha/(1+\alpha)$, $-1<\eta<-2\alpha/(1+\alpha)$, and $-2\alpha/(1+\alpha)<\eta<1$, respectively.

IV. PROBABILITY DENSITY FUNCTION OF THE ABSORPTION TIME

Equation (1.2) describes the temporal evolution of a system with an absorbing state at $x=0$. We are therefore particularly interested in studying the first-passage time to that state, i.e., the absorption time t_0, and how it depends on the characteristics of the random driving force. Since $A(t)$ is the probability that at time t the system is in the absorbing state, the difference $A(t+dt)-A(t)$ is the probability that $x(t)$ reaches that state during the time interval $(t,t+dt)$. Consequently, the PDF $w(t)$ of the absorption time t_0 is given by

$w(t)=dA(t)/dt$. Using Eq. (2.12), we obtain for $w(t)$ the following general expression:

$$w(t)=\sqrt{\frac{2}{\pi}}(1-\alpha)^2 x_0^{1-\alpha}\frac{e^{-\omega t}\Delta_\omega(t)}{\sigma_q^3(t)} \times\exp\left(-\frac{x_0^{2(1-\alpha)}e^{-2\omega t}}{2\sigma_q^2(t)}\right). \quad (4.1)$$

This formula implies that $w(t)$ goes to zero as t goes to zero and as t goes to infinity. The short-time behavior of $w(t)$ reflects the initial condition $x(0)=x_0>0$. The probability that the system enters the absorbing state at time t has to vanish as t approaches zero. The PDF of the absorption time must also approach zero as t goes to infinity, since it is integrable, $\int_0^\infty dt w(t)=A(\infty)$. In other words, conditioned on the sample paths that ultimately become trapped in the state $x=0$, the probability that the first-passage time into that state is t, must go to zero sufficiently fast as t goes to infinity. Note that $w(t)$ is normalized to 1 only in the case of almost sure absorption, $A(\infty)=1$.

If the colored Gaussian noise $f(t)$ is Markovian, i.e., if it is an Ornstein-Uhlenbeck process, then its correlation function has the exponential form $R(u)=R(0)e^{-u/t_c}$, where t_c is the correlation time. In this case, Eqs. (2.6) and (2.8) yield

$$\sigma_q^2(t)=(1-\alpha)^2R(0)t_c^2\frac{e^{-2\lambda\tau}}{\lambda(\lambda^2-1)}[\lambda+1+(\lambda-1)e^{2\lambda\tau} -2\lambda e^{(\lambda-1)\tau}] \quad (4.2)$$

and

$$\Delta_\omega(t)=R(0)t_c\frac{1-e^{-(\lambda+1)\tau}}{\lambda+1} \quad (4.3)$$

($\tau=t/t_c$, $\lambda=\omega t_c$), respectively, and Eq. (4.1) can be written in the form

$$w(t)=-\frac{2}{t_c}\sqrt{\frac{p}{\pi}}\frac{dV_\lambda(\tau)}{d\tau}\exp[-pV_\lambda^2(\tau)]. \quad (4.4)$$

Here

$$p=\frac{x_0^{2(1-\alpha)}}{2(1-\alpha)^2R(0)t_c^2} \quad (4.5)$$

is a dimensionless parameter, and

$$V_\lambda(\tau)=\left(\frac{\lambda(\lambda^2-1)}{\lambda+1+(\lambda-1)e^{2\lambda\tau}-2\lambda e^{(\lambda-1)\tau}}\right)^{1/2} \quad (4.6)$$

is a monotonically decreasing function of τ that at the singular points $\lambda=0,\pm1$ is defined as

$$V_{-1}(\tau)=\left(\frac{2}{1-(1+2\tau)e^{-2\tau}}\right)^{1/2}, \quad (4.7)$$

$$V_0(\tau)=\frac{1}{\sqrt{2}}\left(\frac{1}{e^{-\tau}-1+\tau}\right)^{1/2}, \quad (4.8)$$

$$V_1(\tau)=\left(\frac{2}{e^{2\tau}-1-2\tau}\right)^{1/2}. \quad (4.9)$$

For $\tau\to0$, the function $V_\lambda(\tau)$ has a single asymptotic behavior $V_\lambda(\tau)\sim\tau^{-1}$. For $\tau\to\infty$, Eqs. (4.6)–(4.9) yield different asymptotic behaviors for different values of λ:

$$V_\lambda(\tau)\sim\begin{cases}\sqrt{\lambda(\lambda-1)}\left(1+\frac{\lambda e^{(\lambda-1)\tau}}{1+\lambda}\right), & \lambda<-1,\\ \sqrt{2}(1+\tau e^{-2\tau}), & \lambda=-1,\\ \sqrt{\lambda(\lambda-1)}\left(1+\frac{(1-\lambda)e^{2\lambda\tau}}{2(1+\lambda)}\right), & -1<\lambda<0,\\ 1/\sqrt{2\tau}, & \lambda=0,\\ \sqrt{\lambda(\lambda+1)}e^{-\lambda\tau}, & \lambda>0.\end{cases} \quad (4.10)$$

If $R(0)\to\infty$, $t_c\to0$, such that $R(0)t_c=\Delta$, i.e., if $f(t)$ is Gaussian white noise with intensity Δ, then $x(t)$ is a Markovian diffusion process [28]. Since $\lambda\to0$ and $\tau\to\infty$ ($\lambda\tau=\omega t$), Eqs. (4.2) and (4.3) can be written as

$$\sigma_q^2(t)=(1-\alpha)^2\Delta\,\frac{1-e^{-2\omega t}}{\omega} \quad (4.11)$$

and $\Delta_\omega(t)=\Delta$, respectively. Taking into account also that

$$\lim_{t_c\to0}\frac{V_\lambda^2(\tau)}{t_c}=\frac{\omega}{e^{2\omega t}-1}, \quad (4.12)$$

we obtain from Eq. (4.4) the formula

$$w(t)=\sqrt{\frac{2}{\pi\Delta}}\frac{x_0^{1-\alpha}}{1-\alpha}e^{2\omega t}\left(\frac{\omega}{e^{2\omega t}-1}\right)^{3/2} \times\exp\left(-\frac{x_0^{2(1-\alpha)}}{2(1-\alpha)^2\Delta}\,\frac{\omega}{e^{2\omega t}-1}\right), \quad (4.13)$$

which is valid for all real values of ω. In particular, for $\alpha=\omega=0$ [when $x(t)$ is the Wiener process or Brownian motion] Eq. (4.13) reduces to the known result [29]

$$w(t)=\frac{x_0}{\sqrt{4\pi\Delta}t^{3/2}}\exp\left(-\frac{x_0^2}{4\Delta t}\right). \quad (4.14)$$

V. MEAN ABSORPTION TIME

A. General results

In the previous section, we have derived the PDF of the absorption time, i.e., of the first-passage time of the system from $x(0)=x_0>0$ to $x=0$. An important numerical characteristic of that PDF is the mean first-passage time or mean absorption time T. Since for $\kappa<0$, absorption does not occur almost surely, the mean absorption time is given by a conditional average, namely, $T=\langle t_0\rangle_a$, where $\langle\rangle_a$ denotes averaging with respect to those sample paths of $f(t)$ for which the system state $x(t)$ eventually becomes trapped in the absorbing state $x=0$. Using the general expression (4.1) for the absorption time PDF, we can write the mean absorption time T in the form

$$T=\int_0^\infty dt\, tw(t). \tag{5.1}$$

If the systematic force drives the system toward the absorbing state, i.e., if $\kappa>0$ $(\omega>0)$, then the absorption time PDF decays exponentially, $w(t)\propto e^{-\omega t}$, as $t\to\infty$, and the mean absorption time T is finite, $T<\infty$. If the systematic force drives the system away from the absorbing state, i.e., if $\kappa<0$ $(\omega<0)$, then absorption is not certain, $A(\infty)<1$. As $t\to\infty$, Eqs. (2.8) and (2.19) yield $\Delta_\omega(t)=o(e^{|\omega|t})$ and $\sigma_q^2(t)\propto e^{2|\omega|t}$, respectively. Therefore, $w(t)=o(e^{-|\omega|t})$ as $t\to\infty$, and the mean first-passage time conditioned on absorption is again finite, $T<\infty$. The critical case $\kappa=0$ $(\omega=0)$ is more complicated, since either stochastic localization or various types of diffusive behavior occur. If the condition (2.22) does not hold, then $\sigma_q^2(t)\sim\Delta_0(t)t\to\infty$ and

$$w(t)\sim\sqrt{\frac{2}{\pi}}(1-\alpha)^2\frac{x_0^{1-\alpha}}{\Delta_0^{1/2}(t)t^{3/2}} \tag{5.2}$$

as $t\to\infty$. Since $\Delta_0(t)/t\to0$ for $t\to\infty$, we obtain from Eq. (5.2) that $w(t)t\to0$ and $w(t)t^2\to\infty$, i.e., $T=\infty$. If the condition (2.22) holds, then $\sigma_q(\infty)<\infty$ and Eq. (4.1) yields $w(t)\propto\Delta_0(t)$ for $t\to\infty$. In the case of stochastic localization, $T<\infty$, if $\Delta_0(t)=o(1/t^2)$, and $T=\infty$, if $t^2\Delta_0(t)\to\infty$ as $t\to\infty$.

If $A(\infty)=1$, the sample paths that contribute to T include with probability 1 all sample paths of the random driving force $f(t)$. The mean absorption time T coincides then with the unconditional mean absorption time, which is defined by the averaging of t_0 over all sample paths of $f(t)$, i.e., $T=\langle t_0\rangle$. For $A(\infty)<1$, the last equality can be violated. The total probability of the sample paths for which $x(t)$ does not reach the absorbing state, even as time goes to infinity, is nonzero [it equals $1-A(\infty)$], and so $\langle t_0\rangle=\infty$, while T can be finite as we saw above.

B. Ornstein-Uhlenbeck noise

To gain further insight into the temporal behavior of the system (1.2), we study the properties of the mean absorption time T for two cases, namely, Ornstein-Uhlenbeck noise in this subsection and Gaussian white noise in the next. According to Eqs. (4.4) and (5.1), the mean absorption time for the case of Ornstein-Uhlenbeck noise is given by

$$T=-2t_c\sqrt{\frac{p}{\pi}}\int_0^\infty d\tau\,\tau\frac{dV_\lambda(\tau)}{d\tau}e^{-pV_\lambda^2(\tau)}. \tag{5.3}$$

Introducing the new variable $v=V_\lambda(\tau)$, we can rewrite Eq. (5.3) as

$$T=2t_c\sqrt{\frac{p}{\pi}}\int_{V_\lambda(\infty)}^\infty dv\, e^{-pv^2}\tau_\lambda(v), \tag{5.4}$$

where $V_\lambda(\infty)=\sqrt{\lambda(\lambda-1)}$ for $\lambda<0$, $V_\lambda(\infty)=0$ for $\lambda\geqslant0$, and $\tau_\lambda(v)$ is the solution of the equation $v=V_\lambda(\tau)$ with respect to τ. Since $V_\lambda(\tau)\sim\tau^{-1}$ as $\tau\to0$, we obtain $\tau_\lambda(v)\sim v^{-1}$ as $v\to\infty$, and, using Eq. (4.10), we find the main term of the asymptotic expansion of $\tau_\lambda(v)$ for $v\to V_\lambda(\infty)$:

$$\tau_\lambda(v)\sim\begin{cases}(\lambda-1)^{-1}\ln[v-V_\lambda(\infty)], & \lambda<-1,\\ (2\lambda)^{-1}\ln[v-V_\lambda(\infty)], & -1\leqslant\lambda<0,\\ (2v^2)^{-1}, & \lambda=0,\\ -\lambda^{-1}\ln v, & \lambda>0.\end{cases} \tag{5.5}$$

This result implies that the integral in Eq. (5.4) diverges only for the critical case, i.e., if $\lambda=0$ $(\kappa=0)$. So, $T=\infty$ for $\lambda=0$, and $T<\infty$ otherwise, which agrees with our general results.

For $\lambda\neq0$, we use Eq. (5.4) to determine the asymptotic behavior of T as $p\to0$ and $p\to\infty$. As is clear from Eq. (4.5), physically the limit $p\to0$ corresponds to the limit of either the initial system state x_0 approaching the absorbing state $x=0$ or to the limit of the noise variance $R(0)$ going to infinity or the correlation time t_c going to infinity. In the same way, the limit $p\to\infty$ corresponds to the limit of either $x_0\to\infty$ or $R(0)\to0$ or $t_c\to0$. In the first limit case, $p\to0$, we represent the integral in Eq. (5.4) as a sum of two integrals over disjoint intervals $[V_\lambda(\infty),a]$ and (a,∞). Since for $\lambda\neq0$ and $p\to0$, the first integral converges and the second one diverges, the asymptotic equality

$$\int_{V_\lambda(\infty)}^\infty dv\, e^{-pv^2}\tau_\lambda(v)\sim\int_a^\infty dv\, e^{-pv^2}\tau_\lambda(v) \tag{5.6}$$

holds. If the value of a is large enough, i.e., $\tau_\lambda(a)\sim a^{-1}$, then

$$\int_a^\infty dv\, e^{-pv^2}\tau_\lambda(v)\sim\int_{\sqrt{p}a}^\infty dx\frac{e^{-x^2}}{x}\sim\frac{1}{2}\ln\frac{1}{p} \tag{5.7}$$

$(p\to0)$, and we obtain

$$T\sim t_c\sqrt{\frac{p}{\pi}}\ln\frac{1}{p}\quad(p\to0). \tag{5.8}$$

For $p\to\infty$, the dominant contribution to the integral in Eq. (5.4) comes from the lower limit of integration. In this case, Eqs. (5.4) and (5.5) yield

$$T\sim2t_c\sqrt{\frac{p}{\pi}}a_\lambda\int_{V_\lambda(\infty)}^\infty dv\, e^{-pv^2}\ln[v-V_\lambda(\infty)], \tag{5.9}$$

where $a_\lambda=(\lambda-1)^{-1}$ if $\lambda<-1$, $a_\lambda=(2\lambda)^{-1}$ if $-1\leqslant\lambda<0$, and $a_\lambda=-\lambda^{-1}$ if $\lambda>0$. Introducing the new variable $x=v-V_\lambda(\infty)$ and taking into account that

$$\int_0^\infty dx\, e^{-px^2}\ln x \sim \frac{1}{4}\sqrt{\frac{\pi}{p}}\ln\frac{1}{p} \quad (p\to\infty), \tag{5.10}$$

we find from Eq. (5.9) for $\lambda>0$ that

$$T\sim\frac{t_c}{2\lambda}\ln p \quad (p\to\infty). \tag{5.11}$$

If $\lambda<0$, then $V_\lambda(\infty)\neq 0$, and the asymptotic formula

$$\int_0^\infty dx\, e^{-p[x+V_\lambda(\infty)]^2}\ln x\sim\frac{e^{-pV_\lambda^2(\infty)}}{2pV_\lambda(\infty)}\ln p \tag{5.12}$$

$(p\to\infty)$ holds, and so

$$T\sim t_c|a_\lambda|\frac{e^{-pV_\lambda^2(\infty)}}{\sqrt{\pi p V_\lambda^2(\infty)}}\ln p \quad (p\to\infty). \tag{5.13}$$

If $p\to\infty$, then according to Eqs. (5.11) and (5.13) $T\to\infty$ for $\lambda>0$, and $T\to 0$ for $\lambda<0$. This result, the second part of which is rather puzzling at first sight, can be understood as follows. Recall that $\lambda=(1-\alpha)\kappa t_c$. For $\lambda>0$, the systematic force drives the system toward the absorbing state, while for $\lambda<0$, it drives the system away from that state. That $T\to\infty$ as $p\to\infty$ for $\lambda>0$ then reflects simply the fact that, the farther away the system starts from the absorbing state or the weaker the colored noise, the longer it takes on average for the system to become trapped in the state $x=0$. The seemingly strange results that $T\to 0$ as $p\to\infty$ for $\lambda<0$ can be understood as follows. The systematic force drives the system away from the absorbing state and, as either the initial state of the system goes to infinity or the influence of the noise goes to zero, the probability for the system to be in the absorbing state at $t=\infty, A(\infty)$, rapidly goes to zero. The mean absorption time T is a *conditional* average, and the total probability of sample paths of $f(t)$ that lead to absorption goes to zero. Indeed, for $R(u)=R(0)e^{-u/t_c}$, Eqs. (2.12) and (4.2) yield $A(\infty)=\mathrm{erfc}[\sqrt{p}V_\lambda(\infty)]$, and therefore

$$A(\infty)\sim\frac{e^{-pV_\lambda^2(\infty)}}{\sqrt{\pi p V_\lambda^2(\infty)}} \quad (p\to\infty). \tag{5.14}$$

The comparison of Eqs. (5.13) and (5.14) shows that $T\sim t_c|a_\lambda|A(\infty)\ln p$ and $T\to 0$ as $p\to\infty$.

C. Gaussian white noise

When $f(t)$ is Gaussian white noise, it is easily verified that the intrinsic boundary $x=0$ of the Markovian diffusion process $x(t)$ is accessible in finite time and is a regular boundary [23]. The PDF of the absorption time is given by Eq. (4.13). Substituting it into Eq. (5.1) and introducing the new variable $y=\{\mathrm{sgn}\,\omega/[\exp(2\omega t)-1]\}^{1/2}$, we obtain for $\omega\neq 0$

$$T=\sqrt{\frac{g}{\pi}}\frac{1}{\omega}\int_{I_\omega}^\infty dy\, e^{-gy^2}\ln\left(1+\frac{\mathrm{sgn}\,\omega}{y^2}\right), \tag{5.15}$$

where

$$g=\frac{x_0^{2(1-\alpha)}|\omega|}{2(1-\alpha)^2\Delta} \tag{5.16}$$

is a dimensionless parameter, and

$$I_\omega\equiv\lim_{t\to\infty}\frac{\omega}{|\omega|(e^{2\omega t}-1)}=\begin{cases}0, & \omega>0,\\ 1, & \omega<0.\end{cases} \tag{5.17}$$

(Recall that $T=\infty$ for $\omega=0$.) Taking into account that $I_\omega^2=I_\omega$ and

$$1+\frac{\mathrm{sgn}\,\omega}{I_\omega+x}=\begin{cases}1+1/x, & \omega>0,\\ (1+1/x)^{-1}, & \omega<0,\end{cases} \tag{5.18}$$

we can reduce Eq. (5.15) by a change of variables $x=y^2-I_\omega$ to the form

$$T=\sqrt{\frac{g}{\pi}}\frac{e^{-gI_\omega}}{2|\omega|}\int_0^\infty dx\frac{e^{-gx}}{\sqrt{I_\omega+x}}\ln\left(1+\frac{1}{x}\right). \tag{5.19}$$

For $\omega>0$, the mean absorption time (5.19) can be represented using generalized hypergeometric functions (see the Appendix)

$$T=\frac{\sqrt{\pi g}}{\omega}{}_1F_1\left(\frac{1}{2};\frac{3}{2};g\right)-\frac{g}{\omega}{}_2F_2\left(1,1;\frac{3}{2},2;g\right). \tag{5.20}$$

The same result follows also from the usual approach [24] to the first-passage time problem for Markovian diffusion processes. According to that approach, the unconditional mean absorption time is given by

$$\langle t_0\rangle=2\int_0^{x_0}\frac{dy}{\phi(y)}\int_y^\infty dz\frac{\phi(z)}{B(z)}, \tag{5.21}$$

where $\phi(x)=\exp\{\int^x dx'[2A(x')/B(x')]\}$, $B(x)=2\Delta x^{2\alpha}$, $A(x)=\alpha\Delta x^{2\alpha-1}-\kappa x$. Using the equality

$$\frac{\phi(z)}{\phi(y)}=\left(\frac{z}{y}\right)^\alpha\exp\left[-\frac{\kappa(z^{2(1-\alpha)}-y^{2(1-\alpha)})}{2\Delta(1-\alpha)}\right] \tag{5.22}$$

and the new variables of integration

$$u=\left(\frac{y}{x_0}\right)^{2(1-\alpha)}, \quad v=\left(\frac{z}{x_0}\right)^{2(1-\alpha)}-u, \tag{5.23}$$

we can transform Eq. (5.21) to the form

$$\langle t_0\rangle=\frac{g}{2|\omega|}\int_0^1\frac{du}{\sqrt{u}}\int_0^\infty\frac{dv}{\sqrt{v+u}}e^{-gv\,\mathrm{sgn}\,\omega}. \tag{5.24}$$

In agreement with the general results obtained at the beginning of this section, Eq. (5.24) yields $\langle t_0\rangle=\infty$ for $\omega<0$. If $\omega>0$, then using the standard integrals [30]

$$\int_0^\infty \frac{dv}{\sqrt{v+u}} e^{-gv} = \sqrt{\frac{\pi}{g}} e^{gu}\, \mathrm{erfc}(\sqrt{gu}),$$

$$\int_0^1 \frac{du}{\sqrt{u}} e^{gu}\, \mathrm{erfc}\,(\sqrt{gu}) = -2\sqrt{\frac{g}{\pi}}\, {}_2F_2\left(1,1;\frac{3}{2},2;g\right) + 2\, {}_1F_1\left(\frac{1}{2};\frac{3}{2};g\right), \tag{5.25}$$

we obtain from Eq. (5.24) that $\langle t_0\rangle = T$, where T is given by Eq. (5.20), which also agrees with our general results.

As for the case of Ornstein-Uhlenbeck noise, we study the asymptotic behavior of the mean absorption time T, Eq. (5.19), as either the initial state x_0 goes to zero or the noise intensity Δ goes to infinity, i.e., $g\to 0$, and as either the initial state x_0 goes to infinity or the noise intensity Δ goes to zero, i.e., $g\to\infty$. According to Eq. (5.19), if $g\to 0$, then

$$T \sim \sqrt{\frac{g}{\pi}} \frac{1}{2|\omega|} \int_0^\infty \frac{dx}{\sqrt{I_\omega + x}} \ln\left(1+\frac{1}{x}\right), \tag{5.26}$$

and, taking into account that

$$\int_0^\infty \frac{dx}{\sqrt{I_\omega + x}} \ln\left(1+\frac{1}{x}\right) = \begin{cases} 2\pi, & \omega>0, \\ 4\ln 2, & \omega<0, \end{cases} \tag{5.27}$$

we obtain for $g\to 0$

$$T \sim \frac{\sqrt{\pi g}}{|\omega|} \times \begin{cases} 1, & \omega>0, \\ (2\ln 2)/\pi, & \omega<0. \end{cases} \tag{5.28}$$

If $g\to\infty$, then the main contribution to the integral in Eq. (5.19) comes from a small vicinity of the lower limit of integration. Equation (5.19) yields

$$T \sim \frac{e^{-gI_\omega} \ln g}{2\sqrt{\pi}|\omega|} \int_0^\infty dy\, \frac{e^{-y}}{\sqrt{gI_\omega + y}}, \tag{5.29}$$

and so

$$T \sim \frac{\ln g}{2|\omega|} \times \begin{cases} 1, & \omega>0, \\ e^{-g}/\sqrt{\pi g}, & \omega<0, \end{cases} \tag{5.30}$$

as $g\to\infty$. Thus, if $g\to\infty$ then $T\to\infty$ for $\omega>0$, and $T\to 0$ for $\omega<0$, for the same reasons as in the case of Ornstein-Uhlenbeck noise.

VI. CONCLUSIONS

We have derived exact, analytical expressions for various quantities characterizing the absorption process in a model system with a linear growth term and driven by multiplicative colored Gaussian noise. As mentioned in the Introduction, this model describes certain chemical and biological systems. The absorbing state of the Langevin equation (1.2) represents extinction in those applications. If the growth rate is negative, extinction is inevitable. If the growth rate is positive, long-term survival occurs with nonzero probability, but not with probability 1. In the presence of external noise, survival is never certain. In the critical case of zero growth, the fluctuations drive the system to extinction, with one exception. If regions of positive and negative correlations in the noise cancel each other out, then the system has a chance of avoiding extinction. The absorption, or extinction, process was analyzed in more detail for Ornstein-Uhlenbeck noise, the most common model of colored noise. For systems with negative growth rates, the mean time to extinction depends only weakly, namely, logarithmically, on the noise variance as the latter decreases toward zero. Even a large reduction of the noise variance lengthens the average survival time only moderately. For systems with positive growth rates, the probability of ultimate extinction decreases somewhat faster than exponentially as the variance of the noise decreases toward zero. Here, even a small reduction in the variance of the external noise greatly improves the chance of long-term survival.

The method we developed in this paper for the study of the exactly solvable model (1.2) can be extended to the class of models with an absorbing state that are described by Eq. (1.1). We assume that the functions $F(x)$ and $g(x)$ are such that the Langevin equation

$$[\dot{x}(t) - F(x(t))]g^{-1}(x(t)) = f(t) \tag{6.1}$$

$[x(0)=x_1]$ has a single-valued solution $x(t)$ whose range of values contains the point $x=\tilde{x}$. (Without loss of generality we set $\tilde{x}=0$.) Let $G(x,t)$ be the probability density that $x_+(t)=x$, where $x_+(t)$ is the solution of Eq. (6.1) for $x(0)=+x_0$. If $F(-x)=-F(x)$ and $g(-x)=g(x)$, then $G(-x,t)$ is the probability density that $x_-(t)=x$, where $x_-(t)$ is the solution of Eq. (6.1) for $x(0)=-x_0$. With the help of those densities we can construct the probability density $P(x,t)$ of the solution of Eq. (1.1) with an absorbing state at $x=0$. By analogy with Eq. (2.9) we obtain

$$P(x,t) = G(x,t) - G(-x,t) + A(t)\,\delta(x), \tag{6.2}$$

where

$$A(t) = 1 - \int_0^\infty dx[G(x,t) - G(-x,t)] \tag{6.3}$$

is the probability that the system governed by Eq. (1.1) is in the absorbing state $x=0$ at time t. In other words, the PDF $P(x,t)$ for the system (1.1) is fully defined by the PDF $G(x,t)$ for the system described by Eq. (6.1). This simplifies the problem considerably, because no boundary conditions are needed to find $G(x,t)$. Note also that, if the stationary PDF $G_{st}(x) = G(x,\infty)$ exists, then $G_{st}(-x) = G_{st}(x)$, and the system described by Eq. (1.1) is in the absorbing state $x=0$ with probability 1 at $t=\infty$.

APPENDIX

For $\omega > 0$, we write Eq. (5.19) as

$$T=\sqrt{\frac{g}{\pi}}\frac{1}{2\omega}(K_1-K_0), \tag{A1}$$

where

$$K_\sigma=\int_0^\infty dx\,\frac{e^{-gx}}{\sqrt{x}}\ln(\sigma+x). \tag{A2}$$

If $\sigma=0$, then [30]

$$K_0=-\sqrt{\frac{\pi}{g}}[\gamma+\ln(4g)] \tag{A3}$$

($\gamma\approx 0.5772$ is the Euler constant). Using the integral representation of the degenerate hypergeometric function [31]

$$\Psi(a,c;x)=\frac{1}{\Gamma(a)}\int_0^\infty dy\, e^{-xy}y^{a-1}(1+y)^{c-a-1} \tag{A4}$$

($a>0$), we obtain for K_1

$$K_1=-\sqrt{\pi}\,\frac{\partial}{\partial\rho}\Psi\left(\frac{1}{2},\frac{3}{2}-\rho;g\right)\bigg|_{\rho=0}. \tag{A5}$$

We used the following formula [30] to evaluate the derivative in Eq. (A5):

$$\Psi\left(\frac{1}{2},\frac{3}{2}-\rho;g\right)=\frac{\Gamma(1/2-\rho)}{\Gamma(1/2)}g^{\rho-1/2}{}_1F_1\left(\rho;\frac{1}{2}+\rho;g\right)+\frac{\Gamma(\rho-1/2)}{\Gamma(\rho)}{}_1F_1\left(\frac{1}{2};\frac{3}{2}-\rho;g\right), \tag{A6}$$

where ${}_1F_1(a;b;x)$ is the special case of the generalized hypergeometric function [31]

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x)=\sum_{n=0}^\infty\frac{(a_1)_n\dots(a_p)_nx^n}{(b_1)_n\dots(b_q)_nn!}, \tag{A7}$$

and $(a)_n=\Gamma(a+n)/\Gamma(a)$. Taking into account that

$$\begin{aligned}\frac{\partial}{\partial\rho}{}_1F_1\left(\rho;\frac{1}{2}+\rho;g\right)\bigg|_{\rho=0}&=\sum_{n=1}^\infty\frac{\Gamma(n)g^n}{(1/2)_nn!}\\&=2g\sum_{n=0}^\infty\frac{\Gamma^2(n+1)g^n}{(3/2)_n\Gamma(n+2)n!}\\&=2g\sum_{n=0}^\infty\frac{(1)_n(1)_ng^n}{(3/2)_n(2)_nn!}\\&=2g\,{}_2F_2\left(1,1;\frac{3}{2},2;g\right),\end{aligned} \tag{A8}$$

$$\frac{\partial}{\partial\rho}\frac{\Gamma(1/2-\rho)}{\Gamma(1/2)}\bigg|_{\rho=0}=\gamma+2\ln 2, \tag{A9}$$

$$\frac{\partial}{\partial\rho}\frac{\Gamma(\rho-1/2)}{\Gamma(\rho)}\bigg|_{\rho=0}=-2\sqrt{\pi}, \tag{A10}$$

$\lim_{\rho\to 0}\Gamma(\rho-1/2)/\Gamma(\rho)=0$, $\partial g^{\rho-1/2}/\partial\rho|_{\rho=0}=\ln g/\sqrt{g}$, and ${}_1F_1(0;1/2;g)=1$, we find

$$K_1=2\pi\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};g\right)-2\sqrt{\pi g}\,{}_2F_2\left(1,1;\frac{3}{2},2;g\right)-\sqrt{\frac{\pi}{g}}[\gamma+\ln(4g)]. \tag{A11}$$

Substituting Eqs. (A3) and (A11) into Eq. (A1), we obtain Eq. (5.20).

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